So why do we have Dirac and Proca? What more do they tell us?

To understand this we need to think about degrees of freedom. Typically the # d.o.f. is equal to the dimension of configuration space <u>or</u> half the dimension of phase space.

Example: 1 non-relativistic spin 0 particle is described by: $x(t), y(t), z(t) \Rightarrow$ 3D configuration space

or: $x(t), y(t), z(t), v_x(t), v_y(t), v_z(t)$ 6D phase space

$\Rightarrow$ 3 d.o.f. (3N for N particles)

For fields the counting is more subtle. Technically a field has an ∞ # of d.o.fs. since we must specify the field value at each spacetime point. However we know that the "motion" of particle-like field fluctuations is covered by K-G equation, so we can set that aside and ask how many d.o.fs. per point are left over. For spin 0 scalars the K-G equation does it all.

But what if a particle has nontrivial internal spin? Long ago in a land far away, Eugene Wigner developed a method for identifying the degrees of freedom using some of the deeper elements of group theory (see "Wigner's classification" or "induced representations"). I will try to give you a sense of the result.

The main idea (in physics) is that the "motion" of a particle, captured by the 4-momentum P^μ, partly, characterizes the freedom in specifying what a particle is doing (its d.o.f.s). The question then is "after specifying P^μ, what else can be chosen <u>without</u> changing P^μ."

The good news is that counting d.o.f. is independent of reference frame (as it should be in relativity) so we can use the simplest one.

Consider a particle in its rest frame: $P^\mu_{rest} = \begin{pmatrix} mc \\ 0 \\ 0 \\ 0 \end{pmatrix}$ } The transformations which leave this invariant are clearly 3D rotations leaving $\vec{p} = \vec{0}$ unchanged.

It is not intuitive, but we have just as much freedom left over in any other frame.

The upshot is that to characterize the internal spin states, we get to carry over all of the familiar angular momentum information from 3D NR QM.

So let's review some essential results for angular momentum in 3D NR QM.

First of all, in contrast to $\underbrace{\text{linear momentum}}$, angular momentum is <u>always</u> discrete in QM. recall this is continuous for ∞ dimension and discrete for finite dimensions (like particle in a box)

The reason is that the angle through which we rotate is compact, i.e. $\theta \in (0, 2\pi]$.

For quantized internal spin, the total spin magnitude S^2 is fixed once and for all. But we can dynamically change S_z. This is not true for orbital angular momentum where L^2 and L_z can change.

From QM you know that if: $S \sim 0 \Rightarrow S_z = 0$ 1 d.o.f.

$S \sim 1/2 \Rightarrow S_z = \pm \frac{1}{2}\hbar$ 2 d.o.f.

(actually $S^2 = s(s+1)\hbar^2$) $S \sim 1 \Rightarrow S_z = (-1, 0, +1)\hbar$ 3 d.o.f.

Note: $\underbrace{S^2 > S_z^2 = s^2\hbar^2}$ etc.

We can never have $S = S_z$ since then $S_x = S_y = 0$, but we know these 3 cannot be simultaneously known.

As an example consider A^ν and the Proca equation. Naively A^ν should have 4 real parameters to be specified, but being spin 1, we should really only expect 3 $(+1, 0, -1)$.

$$\partial_\mu F^{\mu\nu} - \left(\frac{mc}{\hbar}\right)^2 A^\nu = 0$$

Go to momentum space via $i\hbar\partial_\mu \to p_\mu$ $\left(\partial_\mu \to -\frac{i}{\hbar} p_\mu\right)$ then:

$$-\frac{i}{\hbar} p_\mu \left(-\frac{i}{\hbar} p^\mu A^\nu + \frac{i}{\hbar} p^\nu A^\mu\right) - \left(\frac{mc}{\hbar}\right)^2 A^\nu = 0 = -\frac{1}{\hbar^2} p_\mu \left(p^\mu A^\nu - p^\nu A^\mu\right) - \left(\frac{mc}{\hbar}\right)^2 A^\nu$$

In the rest frame $p_\mu = (-mc, \vec{0})$ so we have:

$\nu = 0$ $\quad \frac{1}{\hbar^2}(mc)^2 A^0 + \frac{1}{\hbar^2}(mc)^2 A^0 - \left(\frac{mc}{\hbar}\right)^2 A^0 = 0 \Rightarrow A^0 = 0$

$\nu \neq 0$ $\quad \frac{1}{\hbar^2}(mc)^2 A^i - \left(\frac{mc}{\hbar}\right)^2 A^i = 0$

} So we have 3 values of A^i to select

As for spinors ψ and the Dirac equation, we know that naively ψ has 4 complex components or 8 real degrees of freedom. But a spin $\frac{1}{2}$ particle should only have 2 $(\pm\frac{1}{2})$ so the Dirac equation must do a lot of cutting down.

To see this consider the Dirac equation: $\gamma^\mu \partial_\mu \psi + \frac{mc}{\hbar}\psi = 0$ in momentum representation $(i\hbar\partial_\mu \to p_\mu)$

$$-i\gamma^\mu p_\mu \psi + mc\psi = 0$$

In the rest fram $p_\mu = (-mc, 0, 0, 0)$ so this becomes: $i\gamma^0 mc\psi + mc\psi = 0$

$$(i\gamma^0 + 1)mc\psi = 0$$

Recall: $\gamma^0 = -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow i\gamma^0 + 1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$

Then: $(i\gamma^0 + 1)mc\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = mc\begin{pmatrix} \psi_1 + \psi_3 \\ \psi_2 + \psi_4 \\ \psi_1 + \psi_3 \\ \psi_2 + \psi_4 \end{pmatrix} = 0$

$\Downarrow$ $\psi_1 = -\psi_3$, $\psi_2 = -\psi_4$ } leaves 2 complex (4 real) degrees of freedom!

WTF!?

Let's actually solve the Dirac equation for a particle at rest, i.e. $\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial y} = \frac{\partial\psi}{\partial z} = 0 \iff \underbrace{\vec{p} = 0}_{i\hbar\partial_\mu \leftrightarrow p_\mu}$

Then we have: $\gamma^0 \partial_0 \psi + \frac{mc}{\hbar}\psi = 0$

$$\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \frac{\partial\psi}{\partial ct} + \frac{mc}{\hbar}\psi = 0 \qquad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

Or:
$$\left.\begin{aligned} -\frac{i}{c}\frac{\partial\psi_3}{\partial t} + \frac{mc}{\hbar}\psi_1 &= 0 \\ -\frac{i}{c}\frac{\partial\psi_4}{\partial t} + \frac{mc}{\hbar}\psi_2 &= 0 \\ -\frac{i}{c}\frac{\partial\psi_1}{\partial t} + \frac{mc}{\hbar}\psi_3 &= 0 \\ -\frac{i}{c}\frac{\partial\psi_2}{\partial t} + \frac{mc}{\hbar}\psi_4 &= 0 \end{aligned}\right\}$$

4 solutions:
$$\psi^{(1)}_{rest} = e^{-i\frac{mc^2}{\hbar}t}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \psi^{(2)}_{rest} = e^{-i\frac{mc^2}{\hbar}t}\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\psi^{(3)}_{rest} = e^{i\frac{mc^2}{\hbar}t}\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \qquad \psi^{(4)}_{rest} = e^{i\frac{mc^2}{\hbar}t}\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

From QM we expect the time dependence of a state to evolve as $e^{-i\frac{E}{\hbar}t}$, and for particles at rest $E = mc^2$ so $e^{-i\frac{mc^2}{\hbar}t}$ indicates the usual behavior.

What about the solutions w/ $e^{i\frac{E}{\hbar}t}$ time dependence? These correspond to antiparticle states! These were given an interpretation early on by Feynman and Stuckelberg as positive energy states moving backwards in time. This has been given a more modern understanding, but remains a useful idea and is actually implemented in Feynman diagrams. This explains the 4 d.o.f. in the Dirac equation, $\pm\frac{1}{2}$ for each particle and antiparticle

$$\boxed{\begin{aligned} &\text{Note: Recall that } S_z = \frac{\hbar}{2}\begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix} \text{ for 4-comp } \psi \\ &\qquad\qquad\qquad = \frac{\hbar}{2}\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \\ &\text{Then: } S_z\psi^{(1)} = \frac{\hbar}{2}\psi^{(1)}, \quad S_z\psi^{(2)} = -\frac{\hbar}{2}\psi^{(2)} \\ &\qquad\quad S_z\psi^{(3)} = -\frac{\hbar}{2}\psi^{(3)}, \quad S_z\psi^{(4)} = \frac{\hbar}{2}\psi^{(4)} \end{aligned}}$$

So the Dirac equation secretly knows about and describes both particle and antiparticle behavior. We will get a clearer picture of how these are related when we study discrete symmetries.